Group theory and the zometool construction toy

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June 26, 2023

Golden Integers

Let $\tau = \text{Golden Ratio} = (1+\sqrt{5})/2 \approx 1.618$
Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ = the set of integers.
Lets call these numbers:

$$\mathbb{Z}[\tau] = \{n + m\tau \mid n, m \in \mathbb{Z}\}$$

the Golden Integers.

Golden Integer Gap Problem: Starting at 0, suppose the only allowed moves are: One step left or right, of size 1 or τ.
After N steps, how close can you get to 0 without being 0?

After 1 move: $x \in \{\pm 1, \pm\tau\}$
Closest to zero: $1 = \tau^0$

After 2 moves: $x \in \{0,\ \pm(\tau-1),\ \pm 2,\ \pm(\tau+1),\ \pm 2\tau\}$
Closest to zero: $\tau - 1 = \tau^{-1} \approx 0.618$.

After three moves: $x \in \{\ldots, \pm 3\tau\}$
Closest to zero: $2 - \tau = \tau^{-2} \approx 0.382$.

Golden Integers and Fibonacci numbers

$\tau^0 = 1 + 0\tau$

$\tau^1 = 0 + 1\tau$

$\tau^2 = 1 + 1\tau$

$\tau^3 = 1 + 2\tau$

$\tau^4 = 2 + 3\tau$

$\tau^5 = 3 + 5\tau$

$\tau^6 = 5 + 8\tau$

$\dots$

Powers of τ are always of the form $n + m\tau$ where n, m are consecutive Fibonacci numbers!

Explanation: $F_{n+2} = F_{n+1} + F_n$ and $\tau^{n+2} = \tau^{n+1} + \tau^n$.

Negative powers of the Golden Ratio

$$\begin{aligned} \tau^{-1} &= -1 + 1\tau \\ \tau^{-2} &= 2 - 1\tau \\ \tau^{-3} &= -3 + 2\tau \\ \tau^{-4} &= 5 - 3\tau \\ \tau^{-5} &= -8 + 5\tau \\ \tau^{-6} &= 13 - 8\tau \end{aligned}$$

...

Negative powers of τ are also Golden Integers $n + m\tau$
but this time n, m are $\pm$ Fibonacci numbers.

Gap problem for the Problem Session

Let $n, m, N = n + m$ be three consecutive Fibonacci numbers.
For instance, $n = 5$, $m = 8$, $N = 13$.

Prove the Gap Problem during the problem session:
Starting at 0, with $\leq N$ moves (each of size ± 1 or $\pm\tau$),
the closest we can get to 0, without being 0, is:

$$|m - n\tau| \geq \frac{1}{m + n(\tau - 1)} = \frac{1}{m + 0.618n} \geq \frac{1}{m + n} = \frac{1}{N}.$$

Conclusion: If you can reach a Golden Integer x with at most N moves, then either $x = 0$ or the gap $|x|$ is $\geq 1/N$.

Zometool Gap Theorem.
If you connect at most N zometool pieces, then the gap between the start and finish is either exactly 0 or $\geq 0.7/N$.

(This is for the standard blue, yellow, and red zome-struts, with the smallest blue having length 1, the smallest yellow having length $\cos(\pi/6)$, and the smallest red having length $\cos(\pi/10)$.)

Application of the Zometool Gap Theorem

The proof of the Zometool Gap Theorem is similar to the Golden Integer Gap question for the problem session, except that we have to consider 3 dimensions instead of 1.

The gap theorem says that, in zome constructions that don't easily bend, either:

- a gap is so large that it is obviously non-zero
- or it is a mathematically exact fit.

Application: Can view zometool constructions in this talk as mathematically exact proofs!

But first, what is zometool?

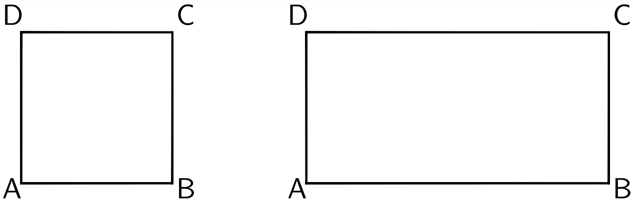
What is the zometool construction toy?

Short answer:

- The group A_5 is the group of rotational symmetries of the icosahedron.
- There is only one good way to design a construction toy with this symmetry group.

To explain this answer, I'll first explain what a symmetry group is.

Group of Symmetries



A group is a concept in algebra in mathematics. In this talk, we only consider groups whose elements are rotations.

Which rotations send the square to the square?
Which rotations send the rectangle to the rectangle?

$$G_{\text{square}} = \{\mathrm{Rot}_0,\ \mathrm{Rot}_{90},\ \mathrm{Rot}_{180},\ \mathrm{Rot}_{270}\}$$

$$G_{\text{rectangle}} = \{\mathrm{Rot}_0,\ \mathrm{Rot}_{180}\}$$

Example: Rot_{90} acts like this $A \mapsto B \mapsto C \mapsto D \mapsto A$.
It has order 4 which means $(\mathrm{Rot}_{90})^4 =$ the identity.

What is a group?

Example:

$$G_{\mathrm{square}} = \{\mathrm{Rot}_0,\ \mathrm{Rot}_{90},\ \mathrm{Rot}_{180},\ \mathrm{Rot}_{270}\}$$

Saying that this is a group means is that when you compose elements, the result is another element.

For instance, composing Rot_{90} with itself gives Rot_{180}.

Composing Rot_{180} with itself gives $\mathrm{Rot}_{360} = \mathrm{identity} = \mathrm{Rot}_0$.

Previous slide: square $\rightsquigarrow$ a group

Converse direction works as well: Pick a point $\neq 0$, apply all rotations from that group, then you get the vertices of a square!

Group of rotational symmetries of a cube

Let G_{cube} be the group of rotations that send a cube to a cube.
Visual demonstration: G_{cube} contains 24 rotations.

There are:

1 rotation of order 1 (rotation of 0 degrees; the identity)

9 rotations of order 2 (rotations of 180 degrees)

8 rotations of order 3 (rotations of 120 or 240 degrees)

6 rotations of order 4 (rotations of 90 or 270 degrees)

Like a cube, ordinary buildings have many 90 degree angles, between walls, floors, struts, etc.

What if we based construction on another symmetry group?

Apart from a cube, what other symmetrical solids are there?

Platonic solids

1. cube (symmetry group $= S_4$, has 24 elements)
2. octahedron (symmetry group $= S_4$)
3. icocahedron (symmetry group $= A_5$, has 60 elements)
4. dodesahedron (symmetry group $= A_5$)
5. tetrahedron (symmetry group $= A_4 \subset A_5$, has 12 elements).

So lets look at A_5! Apart from the identity, it has:
15 order-2-rotations, 20 order-3-rotations, 24 order-5-rotations.

If $n > 1$ then an order-n-rotation has

1. an axis of rotation
2. an angle $\frac{k}{n} \cdot 360$ with $1 \leq k < n$ and $\gcd(k, n) = 1$.

A_5 has $\frac{15}{1} + \frac{20}{2} + \frac{24}{4}$ axes of rotation.

Blue, yellow and red zome-struts align with these rotation axes.

Zome balls, the connectors in zometool construction

Let A_5 = all rotations for the dodecahedron. Take a sphere, a ball, and apply all these rotations to that ball.

- Order-2-rotations. 15 axes of rotation. For each, drill holes in opposite sides of the ball that the axis of rotation fits through.
- Order-3-rotations. 10 axes of rotation. Drill two holes for each of these axes as well.
- Order-5 rotations. 6 axes of rotation. Drill two holes for each of these axes as well.

To be A_5-symmetric, a hole in an order-n-rotation axis must be order-n symmetric. So we must do this:

- Order-2 axes: Drill rectangular holes.
- Order-3 axes: Drill triangular holes.
- Order-5 axes: Drill pentagons.

Next we will look at struts to fit in those holes.

Blue space

The struts in zome that fit the rectangular holes are colored blue, and come in three lengths, b_1, b_2, and b_3.
On this page, lets only use b_1 and use it as unit length $|b_1| = 1$.

If we use b_1 struts in only x, y, z-directions,
then set of reachable points is $\mathbb{Z}^3 \subset \mathbb{R}^3$.

If we use b_1 struts with all order-2 directions, then the set of reachable points is a $\mathbb{Z}[\tau]$-module B_1 which means:

$$v \in B_1 \Longrightarrow \tau v \in B_1$$

Moreover,

$$\mathbb{Z}[\tau]^3 \subsetneq B_1 \subsetneq \frac{1}{2}\mathbb{Z}[\tau]^3.$$

The latter means that if $v \in B_1$ then the coordinates of $2v$ are Golden Integers. These facts can be computed, or obtained from the "Atlas of finite group representations".

Blue space

That B_1 is a $\mathbb{Z}[\tau]$-module means that τb_1 is in B_1, so it's reachable with b_1 struts.

So if we make $b_2 = \tau b_1$ then b_1, b_2 struts fit well together.

Likewise take $b_3 = \tau b_2 = \tau^2 b_1$.

With these choices, if:

B_1 is the set of reachable points with b_1-struts, and
B is the set of reachable points under all blue struts,

then $B_1 = B$.

Index

The index measures how likely a random element of one set will be in the other set.

Lego bricks example: With length-2 bricks we can cover distances $D_2 = \{0, 2, 4, 6, 8, \ldots\}$ and with length-1 bricks we can cover distances $D_1 = \{0, 1, 2, 3, 4, \ldots\}$.

The index of D_2 in D_1 is 2 which means: a random element of D_1 has a $1/2$ chance of being in D_2. That makes it is easy to fit these bricks together in a Lego house.

Lego houses are not built from length-13 and length-17 bricks because (high index) those don't often fit together.

Blue struts b_1, b_2, b_3 fit very well together. What about yellow struts (order-3 rotations) and red struts (order-5 rotations)?

Yellow Space and Red Space

Let y_1 be the shortest yellow. Because $b_2 = \tau b_1$ and $b_3 = \tau^2 b_1$, lets do the same for yellow and red

$$y_2 = \tau y_1, \quad y_3 = \tau^2 y_1, \quad r_2 = \tau r_1, \quad r_3 = \tau^2 r_1$$

Now let Y be the set of all points reachable with yellow struts, and let R be the set of all points reachable with red struts.

Remains to choose: how long should y_1 and r_1 be?

Minimizing the index gives the best fit.

Then $|y_1| = \cos(\pi/6) \approx 0.866$ (or a τ-power times this).
And $|r_1| = \cos(\pi/10) \approx 0.951$ (or a τ-power times this).

These choices give the best fits, they minimizes every index in:

$$\mathbb{Z}[\tau]^3 \subsetneq B_1 = B \subsetneq Y = R \subsetneq \frac{1}{2}\mathbb{Z}[\tau]^3$$

Zome space

If Z is the set of all points reachable with all blue, yellow, red struts, then

$$Y = R = Z$$

and $B \subsetneq Z$ with index 4, which is optimal.

This low index is why zome pieces are so easy to fit together, if you pick struts randomly, there is a $1/4$ chance it'll fit!

Demonstration with zome: if we have a zome-construction, and want to add a connection between two vertices, then if we choose struts randomly, the chance of success is $1/4$.

Can we improve that?

Labeling elements of zome space

Fix some $v \in Z$ with $v \notin B$. Then

$$Z = Z_0 \bigcup Z_1 \bigcup Z_\tau \bigcup Z_{\tau+1}$$

where

$$Z_\ell = \ell v + B$$

This "coset" Z_ℓ is the set of all vectors that you can write as ℓ times v plus some element of B.

If $p \in Z$ then the label of p is the element $\ell \in \{0, 1, \tau, \tau + 1\}$ for which $p \in Z_\ell$.

Adding labels

For any $p \in Z$ we have $p + p \in B$ (see demonstration). Elements of B have label 0. So the sum of a label plus itself is always 0.

This means that if you add labels, you work modulo 2.

Example: Suppose that p_1, p_2, p_3 have label $\tau + 1$, $\tau + 1$, and τ.
Then $p_1 + p_2 + p_3$ has label $3\tau + 2$.
Modulo 2, even numbers reduce to 0 and odd numbers to 1.
So the label of $3\tau + 2$ reduces to τ in this example.

Labels can be multiplied by τ (they form a "$\mathbb{Z}[\tau]$-module").
For example, if p_1 has label $\tau + 1$ then τp_1 has label
$\tau(\tau + 1) = \tau^2 + \tau = \tau + 1 + \tau$ which reduces to 1.

Vectors $p_1, p_2, \ldots$ form a closed loop if they add to 0. Then their labels add to 0 as well:

In a closed loop the labels add to 0.

Labeling zome struts

A_5 cannot non-trivially permute 4 objects (this is the reason why there is no general formula for solving degree 5 equations!).

This implies that A_5 rotations do not change labels!
So zome struts have labels, independent of their direction.

We can choose the label 1 for y_1. Then:

Blue struts: label 0

y_1 struts: label 1
y_2 struts: label τ
y_3 struts: label $\tau + 1$

r_1 struts: label $\tau + 1$ (demonstration in zome)
r_2 struts: label 1
r_3 struts: label τ.

Label theorem

With a random sum of yellow and red struts, the chance it will fit with blue struts is $1/4$, because the index of $B \subsetneq Z$ is 4.

There are 4 labels. So if we take struts with the correct label-sum, the chance that our construction will fit increases a factor 4.

Label theorem Take $n \geq 3$ zome-struts. All of these n struts fit in an exact closed loop if and only if:

- The longest of the n struts is shorter than the sum of the other $n-1$ struts combined.
- The same also holds for the conjugate-lengths.
- The sum of the labels is 0.

(only struts of types $b_1, b_2, b_3,\ y_1, y_2, y_3,\ r_1, r_2, r_3$ are considered here)

Conjugate lengths

$$\mathbb{Z}[\tau]^3 \subsetneq B \subsetneq Z \subsetneq \frac{1}{2}\mathbb{Z}[\tau]^3 \subsetneq \mathbb{Q}[\sqrt{5}]^3$$

Conjugation means: replace every $\sqrt{5}$ by $-\sqrt{5}$. Notation $v \mapsto \overline{v}$. Sends a closed loop to a closed loop. So "longest < sum rest" must also be true after conjugation!

$|b_1|^2 = 1$. Conjugate = same. So $|\overline{b_1}| = |b_1|$.

$|y_1|^2 = 3/4$. Conjugate = same. So $|\overline{y_1}| = |y_1|$.

$|r_1|^2 = \frac{1}{8}(5+\sqrt{5}) \mapsto \frac{1}{8}(5-\sqrt{5})$. Then $|\overline{r_1}| = |r_0| = |\tau^{-1} r_1|$.

$|\overline{\tau}| = |\tau^{-1}|$ so longer struts have shorter conjugate-lengths! Conjugates times τ^2 are:

$$b_1 \mapsto b_3, \quad b_2 \mapsto b_2, \quad b_3 \mapsto b_1$$
$$y_1 \mapsto y_3, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_1$$
$$r_1 \mapsto r_2, \quad r_2 \mapsto r_1, \quad r_3 \mapsto r_0.$$

Conjugate lengths

If $v \in \mathbb{Z}[\tau]$ then is not 0, then $v\overline{v}$ is a non-zero integer, so

$$|\overline{v}| \cdot |v| \geq 1$$

So the closer $|v|$ is to 0, the further $|\overline{v}|$ must be away from 0.

The first step to prove the Zometool Gap Theorem is: bound how long $\overline{v}$, a sum of N conjugate-lengths, can be. Since the bound is a constant times N, we find $|v| \geq$ a constant divided by N.

The constant depends on how well we bound the relation between $|v|$ and the coordinates of v in $\frac{1}{2}\mathbb{Z}[\tau]$.