Mathematical Summer in Paris 2023
Problems for the lecture by Mark van Hoeij

In these problems, we will use the following notations:

- The *Fibonacci sequence* is the sequence $(F_n)_{n\geq 0}$ defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.
- The *golden ratio* is $\tau = \frac{1+\sqrt{5}}{2}$.

Problem 1. Verify that $\tau^2 - \tau - 1 = 0$.

Problem 2. For any integer $n \geq 2$, prove that $\tau^n = F_{n-2} + F_{n-1}\tau$. Also, for any integer $n \geq 1$, prove that $\tau^{-n} = (-1)^n(F_n - F_{n-1}\tau)$.

Problem 3. Show that the limit of $\frac{F_{n+1}}{F_n}$ as n goes to infinity is the golden ratio τ.

Problem 4. Show that two consecutive Fibonacci are always coprime (that is, their greatest common divisor is always 1).

Problem 5. For $n = 1, 2, 3, 4, 5$, compute the remainder of the Euclidean division of F_n by F_{n-1}. What do you notice? State a conjecture, and try to prove it.

Problem 6. The *Gap Problem* can be stated as follows: you start at zero, and you move at most $N \geq 1$ steps, each step being of size ± 1 or $\pm\tau$. What is the closest you can get to zero without getting back zero?

In other words: what is the smallest non-zero value of $|m + n\tau|$ if m and n are integers such that $|m| + |n| \leq N$?

Prove that this value is at least $\frac{1}{N}$. (One idea: first prove that you can assume that m and n have different signs. Then, see what happens when you multiply $(m - n\tau)$ by $(m + n\tau^{-1})$.)

Problem 7. Prove the following exact solution of the Gap Problem: the smallest non-zero value of $|x + y\tau|$, if x and y are integers such that $|x| + |y| \leq N$, is given as follows: if n is the largest non-negative integer such that $F_{n+1} \leq N$, then $x = F_n$ and $y = -F_{n-1}$.

Problem 8. This problem is about the Euclidean algorithm to compute the gcd of two integers. *If you're unfamiliar with the Euclidean algorithm, go read about it and come back to this problem!*

(1) Show that the number of iterations of the Euclidean algorithm needed to compute the gcd of F_{n+1} and F_n is precisely n.
(2) Let $x > y > 0$ be integers. Assume that the Euclidean algorithm applied to x and y stops after n steps. Prove that $x \geq F_{n+1}$ and $y \geq F_n$.
(3) ($\star$) Deduce from the above that, with the above assumptions, the number of steps after which the Euclidean algorithm stops is in $O\big(\log_\tau(y)\big)$. This result is due to Léger (1837) and Lamé (1844).